Graphs, Frobenius functionals, and the classical Yang-Baxter equation

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Abstract. A Lie algebra $\mathfrak{g}$ is Frobenius if it admits a linear functional $F \in \mathfrak{g}^*$ such that the Kirillov form $B_F(x,y) = F([x,y])$ is non-degenerate. If $\mathfrak{g}$ is the mth maximal parabolic subalgebra $\mathcal{P}(n,m)$ of $\mathrm{sl}(n)$ this occurs precisely when $(n,m) = 1$. We define a *cyclic* functional F on $\mathcal{P}(n,m)$ and prove it is non-degenerate using properties of graphs associated to F. These graphs also provide in certain cases readily computable associated solutions of the classical Yang-Baxter equation. We define the *full local ring* of a graph from which we show that the graph can be reconstructed (as well as a *reduced* local ring), consider the seaweed algebras of Dergachev and Kirillov, and examine the degeneration of solutions to the modified classical Yang-Baxter equation.

1 Introduction

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field K and $F \in \mathfrak{g}^*$ be a linear functional. The associated skew bilinear *Kirillov form* B_F is defined by $B_F(x,y) = F([x,y])$ for $x, y \in \mathfrak{g}$. In this paper K will have characteristic zero although some important assertions hold more generally. The *index* of $\mathfrak{g}$, $\mathrm{idx}(\mathfrak{g})$, is the minimum dimension of $\ker B_F$ as F ranges over $\mathfrak{g}^*$. Those F for which the minimum is achieved are called *regular* and form a Zariski open and dense subset $\mathfrak{g}^*_{\mathrm{reg}}$ of $\mathfrak{g}^*$. Clearly $\mathfrak{g}$ operates on $\mathfrak{g}^*_{\mathrm{reg}}$; if $F \in \mathfrak{g}^*, x \in \mathfrak{g}$ then $[x,F](y) = F([x,y])$ for all $y \in \mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is *Frobenius* if its index is zero, i.e., if there exists an $F \in \mathfrak{g}^*$ such that B_F is non-degenerate. We will call such an F a *Frobenius functional* and denote Frobenius Lie algebras by $\mathfrak{f}$.

Frobenius Lie algebras are intimately connected to skew solutions of the classical Yang-Baxter equation (CYBE). An element $r \in \mathfrak{g} \wedge \mathfrak{g}$ is a solution to the CYBE if $[r,r] = 0$, where $[-,-]$ is the Schouten bracket on $\bigwedge^* \mathfrak{g}$. For any Lie algebra $\mathfrak{g}$ and $F \in \mathfrak{g}^*$ the bilinear form B_F is by definition a coboundary in the Chevalley-Eilenberg complex. Belavin and Drinfel'd call $\mathfrak{g}$ *quasi-Frobenius*

if it admits a non-degenerate skew bilinear form B (not necessarily of the form B_F) which is a $2-$cocycle. For such Lie algebras they note that if B_{ij} is the matrix of B relative to some basis $x_1, \ldots, x_d$ of $\mathfrak{f}$ then $\sum (B^{-1})_{ij} x_i \wedge x_j$ is a skew solution to the classical Yang-Baxter equation (CYBE), [2], [3].

For applications, it is useful to have explicit functionals F so that the r-matrix can be exhibited. Our main interest here is $\mathcal{P}(n,m)$, the mth maximal parabolic subalgebra of $\mathrm{sl}(n)$ (the set of all $n \times n$ matrices X of trace zero with $x_{ij} = 0$ whenever both $i \geq m+1$ and $j \leq m$). A theorem of Elashvili asserts in particular that it is Frobenius if and only if $(m,n) = 1$, [5] (where it is noted that a partial result appears in Rais [13], but this part covers the case here only of $m = 1$.) While any generic $F \in \mathcal{P}(n,m)^*$ will be Frobenius, such functionals are not feasible for computations.

In this note we use graphical methods to show that a canonically defined "cyclic" functional on $\mathcal{P}(n,m)$ is Frobenius, and that in many cases the associated r-matrix is readily computable; these results are proved in Sections 3–8. The cyclic functional is determined by writing the integers $\{1, 2, \ldots, n\}$ in the cyclic order $\{1, m+1, 2m+1, \ldots, (n-1)m+1\}$, where each element greater than m is reduced modulo m. Our computations make substantial use of the "principal element" of $\mathfrak{f}$, which was introduced in [6] and is reviewed in Section 2. Other examples of Frobenius functionals on $\mathcal{P}(n,m)$ are discussed, as well as the "seaweed" algebras of Dergachev-Kirillov [4] (called "biparabolic" by A. Joseph [9]); see Sections 9–10.

In Section 11 we revisit some comments of [8] pertaining to degenerations of solutions to the modified classical Yang-Baxter equation (MCYBE). An element $r \in \mathfrak{g} \wedge \mathfrak{g}$ is a solution to the MCYBE if $[r,r]$ is a non-zero invariant element of $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$. In [8] we stated without proof a remark about the moduli space of solutions to the MCYBE which asserted, in effect, which ones were limits of others in that they lie in boundaries of their orbits, but the proof was omitted. This is quite easy to demonstrate for $\mathrm{sl}(n)$ and we do so here. There seems, however, to be a relation between this and properties of the principal element associated to the "cyclic" functional which we define. Lastly, in Section 12, we define two local rings associated to a graph which, although not used in our main results, might be of independent interest. The "full" local ring enjoys the property that it completely characterizes the graph, with just one exception.

2 The principal element

In this section we recall some results of [6] which will be needed throughout the paper. Let F be a Frobenius functional on a Lie algebra $\mathfrak{f}$. The natural map $\mathfrak{f} \to \mathfrak{f}^*$ defined by $x \mapsto F([x,-])$ is then invertible; the image of F under the inverse is called the *principal element* of $\mathfrak{f}$ and will be denoted $\hat{F}$. It is the unique element of $\mathfrak{f}$ such that $F([\hat{F}, x]) = F(x)$, or $F \circ \mathrm{ad}\, \hat{F} = F$; it depends, of course, on the choice of Frobenius functional. Let $\mathfrak{G}$ be the adjoint algebraic group of $\mathfrak{f}$, i.e., the smallest algebraic Lie group whose Lie algebra contains $\mathrm{ad}\, \mathfrak{g}$. Then $\mathfrak{G}$ operates on $\mathfrak{f}^*$ and the subset of Frobenius functionals is stable under

this action but we will see that the action need not be transitive.

When $\mathfrak{f}$ is a subalgebra of a simple Lie algebra $\mathfrak{g}$ we will say that $\mathfrak{f}$ is *saturated* if it is not an ideal of any larger subalgebra of $\mathfrak{g}$ (e.g., if it contains a Cartan subalgebra of $\mathfrak{g}$). In that case, $\hat{F}$ is semisimple, see Theorem 1 of [6]. (It need not be in general; for a simple example cf. Ooms [10].) For $\mathfrak{g} = \mathrm{sl}(n)$, Theorem 2 of [6] asserts that the eigenvalues of $\mathrm{ad}\,\hat{F}$ must all be integers and independent of the choice of Frobenius functional F. (Dergachev has communicated to the authors that this holds more generally for algebraic $\mathfrak{g}$.) The proof of Theorem 2 in [6] actually shows that in this case the eigenvalues of $\hat{F}$ are integers and constitute a single unbroken string, i.e., if $i < j$ are eigenvalues then so is any k with $i < k < j$. Under the conditions above (which are probably too restrictive) one sees that no eigenvalue of $\hat{F}$ can be larger than n, so the eigenvalues are bounded and there are, up to similarity, only a finite number of possibilities for $\hat{F}$. In a communication to the authors Dergachev has shown that the eigenvalues of $\hat{F}$ do not determine $\mathfrak{f}$ (contrary to a conjecture in [6]) but it may still be the case that for any given n there are only a finite number of Frobenius subalgebras of $\mathrm{sl}(n)$. Denoting the λ-eigenspace of $\mathfrak{f}$ by f_λ note that since $\dim \mathfrak{f}_\lambda = \dim \mathfrak{f}_{1-\lambda}$ one has also that $\mathrm{tr}(\mathrm{ad}\,\hat{F}) = \frac{1}{2}\dim \mathfrak{f}$, [10]. A method of calculating the principal element for certain Frobenius fuctionals will be given in the next section.

3 The graph $\gamma(S)$

When $\mathfrak{f}$ is Frobenius the eigenvalues of $\hat{F}$ can be computed from any regular $F \in \mathfrak{f}^*$ (in particular, from the generic element), but for $\mathfrak{f} \subset \mathrm{sl}(n)$ it is most convenient to do so from a "small" Frobenius functional [6]. Let e_{ij} denote the $n \times n$ matrix with 1 in the (i,j) place and zeros elsewhere. If S is a subset of the indices $(i,j), i \leq i,j \leq n$ then F_S will denote the functional $\sum_{s\in S} e_s^*$. It is defined on the space M_n of all $n \times n$ matrices but will tacitly be restricted, without change in notation, to any Lie subalgebra $\mathfrak{g}$ of M_n we are considering, it being understood then that those e_{ij} with $(i,j) \in S$ lie in $\mathfrak{g}$. We will say that S *carries* or *supports* F_S. The *directed graph of the functional*, $\gamma(S)$, has vertices the integers $1,\ldots,n$ with an arrow from i to j whenever $(i,j) \in S$. We will call F_S *small* if $\gamma(S)$ is a tree (by definition connected, hence having all the integers $1,\ldots,n$ as vertices and having exactly $n-1$ links). One can show, again by the arguments of [6], that for the maximal parabolic subalgebras of $\mathrm{sl}(n)$ if $\#S < n-1$ then F_S can not be Frobenius.

An important class of functionals on the seaweed subalgebras $\mathfrak{g}$ of $\mathrm{sl}(n)$ was given by Dergachev and Kirillov [4]. These subalgebras, which include the maximal parabolic ones, are discussed later. The Dergachev-Kirillov functionals have the form F_S for some S and are always regular; when $\mathfrak{g}$ is Frobenius they are small Frobenius functionals. The 'meander' introduced in [4] is just $\gamma(S)$. For the maximal parabolic subalgebras of $\mathrm{sl}(n)$ we construct some other Frobenius functionals, principally the "cyclic" functionals, for which the associated solution of the CYBE can sometimes be effectively calculated. This will involve some elementary properties of graphs, next section. We conjecture that every

saturated Frobenius subalgebra of $\mathrm{sl}(n)$ has a small Frobenius functional.

Suppose again that we have a small linear functional F_S on $\mathrm{sl}(n)$ supported by a set S. As in [6], set $\varepsilon_i = e_{ii} - 1/n$. These all have trace zero, one has $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n = 0$, and any $n-1$ of them will serve as a basis for the Cartan subalgebra $\mathfrak{h}$; we generally use $\varepsilon_1, \ldots, \varepsilon_{n-1}$. Denote by $K(S)$ the space spanned over the field K by the e_s with $s \in S$. The space spanned by the ε_i is then dual to $K(S)$ under the bilinear form B_F. One can exhibit explicitly the dual basis to the $e_s, s \in S$: Note that if $s = (i,j) \in S$ then removing the edge $i \to j$ disconnects $\gamma(S)$ so every vertex remains connected precisely to one of i and j. Let d_s be the sum of all those ε_k where k remains connected to i or equivalently, the negative of the sum where k remains connected to j. (If n is connected to i, the former will involve ε_n and the letter will not; similarly if n is connected to j.) Then $B(d_s, e_{s'}) = \delta_{s,s'}$, [6]. The d_s are linearly independent but somewhat more is the case. The directed graph γ_S defines a partial order on the set $\{1, \ldots, n\}$. Conjugating by a suitable permutation matrix we may assume that n is a terminal vertex of γ_S and that the partial order is compatible with the natural order. The d_s, which can now simply be numbered as $d_1, \ldots, d_{n-1}$, by their construction have the property that each d_i is a linear combination only of those ε_j with $j \le i$, with the coefficient of ε_i equal to 1. Thus the linear transformation giving the d_s in terms of $\varepsilon_1, \ldots, \varepsilon_{n-1}$ in fact has determinant equal to 1. The d_s span the Cartan subalgebra $\mathfrak{h}$ of diagonal traceless matrices of $\mathrm{sl}(n)$. Set $D_S = \sum_{s \in S} d_s$. Then one has $[D_S, e_s] = e_s$ for all $s \in S$. The eigenvalues of $\mathrm{ad}(D_S)$ on M_n (hence also on $\mathrm{sl}(n)$) are necessarily integers.

To illustrate this, below in diagrammatic form are two small Frobenius functionals on $\mathcal{P}(7,3)$, the "cyclic" one of § 7 (where it will be shown to be Frobenius), and that of Dergachev-Kirillov (which is known to be Frobenius, [4]). In the two diagrams "**x**" marks the matrix entries which support the functional, light dots mark the places where the matrix entries must be zero, and the diagonal is marked visually by dark dots. The Dergachev-Kirillov functional is constructed by placing **x**s on antidiagonals (lines along which $i+j$ is constant) starting at the corners and proceeding until one reaches the main diagonal. The rank of S_{cyclic} (replace each **x** by 1 and all other entries by 0) is four while that of S_{DK} is five, so they cannot be conjugates.

$$S_{\mathrm{cyclic}} = \begin{pmatrix} \bullet & \circ & \circ & \mathbf{x} & \circ & \circ & \circ \\ \circ & \bullet & \mathbf{x} & \circ & \mathbf{x} & \circ & \circ \\ \mathbf{x} & \circ & \bullet & \circ & \circ & \mathbf{x} & \circ \\ \cdot & \cdot & \cdot & \bullet & \circ & \circ & \mathbf{x} \\ \cdot & \cdot & \cdot & \circ & \bullet & \circ & \circ \\ \cdot & \cdot & \cdot & \circ & \circ & \bullet & \circ \\ \cdot & \cdot & \cdot & \circ & \circ & \circ & \bullet \end{pmatrix}, \quad S_{\mathrm{DK}} = \begin{pmatrix} \bullet & \circ & \circ & \circ & \circ & \circ & \mathbf{x} \\ \circ & \bullet & \circ & \circ & \circ & \mathbf{x} & \circ \\ \mathbf{x} & \circ & \bullet & \circ & \mathbf{x} & \circ & \circ \\ \cdot & \cdot & \cdot & \bullet & \circ & \circ & \circ \\ \cdot & \cdot & \cdot & \circ & \bullet & \circ & \circ \\ \cdot & \cdot & \cdot & \circ & \mathbf{x} & \bullet & \circ \\ \cdot & \cdot & \cdot & \mathbf{x} & \circ & \circ & \bullet \end{pmatrix}$$

The graph $\gamma(S_{\mathrm{DK}})$ is a chain if one disregards the direction of the arrows; this holds for all the Dergachev-Kirillov functionals of [4].

$$\gamma(S_{\mathrm{DK}}): \quad 2 \to 6 \to 5 \leftarrow 3 \to 1 \to 7 \to 4$$

By contrast, the graph $\gamma(S_{\rm cyclic})$ is a rooted tree with all arrows directed away from the root (which in this example is 2); this holds for all the cyclic functionals defined in Section 7.

$$\begin{array}{rccccccc} & 2 & \to & 3 & \to & 6 & & \\ \gamma(S_{\rm cyclic}): & \downarrow & & \downarrow & & & & \\ & 5 & & 1 & \to & 4 & \to & 7 \end{array}$$

One can compute D_S (which turns out to be the principal element $\hat{F}$) and its eigenvalues directly from these graphs. Suppose we want the coefficient of ε_i. Removing a single link from the graph disconnects it and leaves i either connected to or disconnected from n (which here is 7). Removing one link at a time, the coefficient of ε_i =(#times i remains connected to n)−(#times i is disconnected from n). In the examples (calculating without the use of ε_7) we have

$$D_{S_{\rm DK}} = {\rm diag}(1,\ 3,\ 2, -1,\ 1,\ 2,\ 0) - \ (8/7)\,\mathbf{I}_7 \tag{1}$$

$$D_{S_{\rm cyclic}} = {\rm diag}(\ 2,\ 4,\ 3,\ 1,\ 3,\ 2,\ 0) - (15/7)\,\mathbf{I}_7 \tag{2}$$

where $\mathbf{I}_7$ is the 7×7 unit matrix. These are conjugate within the parabolic subgroup of the special linear group SL(7) corresponding to removal of the 3rd negative root. All the $e_{ij}, i, j \le n$ are eigenvectors for ${\rm ad}\, D_S$. To calculate the eigenvalue from $\gamma(S)$, note that there is a unique path on the graph from i to j; the eigenvalue is (# arrows traversed in the direction of the arrow) − (# arrows traversed in the reverse direction). The eigenvalues (counted with their multiplicities) of $D_{S_{\rm DK}}$ and $D_{S_{\rm cyclic}}$ are obviously the same; more important, they are the same on the subset consisting of those e_{ij} which lie in $\mathcal{P}(7,3)$, as we know must be the case in general. (An alternative and frequently easier calculation of D_S: build a diagonal matrix ${\rm diag}(c_1, \ldots, c_n)$ by setting $c_1 = 0$ and defining the remaining entries by requiring that $c_i - c_j = 1$ whenever $i \to j$ in $\gamma(S)$, then subtract a suitable multiple of the identity to reduce the trace to zero.)

When F is Frobenius D_S is its principal element $\hat{F}$, [6]. One can verify here directly Ooms' observation that the sum of its eigenvalues on $\mathcal{P}(7,3)$ must be $(1/2)\dim \mathcal{P}(7,3) = 18$. Since the eigenvalue on e_{ij} is the negative of that on e_{ji} it is sufficient to sum the eigenvalues on the 12 of those $e_{ij} \in \mathcal{P}(7,3)$ whose transposes are not in $\mathcal{P}(7,3)$. Using the cyclic functional and going by rows, these are $1, -1, 0, 2; 3, 1, 2, 4; 2, 0, 1, 3$.

Although we assume throughout that K is a field of characteristic zero, in some places it is sufficient that it be a commutative unital ring where, as in the discussion above, when considering ${\rm sl}(n)$ one must assume further that $1/n$ is invertible.

4 Matching number and index of a graph

To prove that the functionals we define are Frobenius it is useful to have some elementary observations about graphs. (There are many treatises; a brief review

of the concepts relevant here, with references, can be found, for example, in the Wikipedia article [1] and [12].) All our graphs will be finite, not necessarily connected, with any two vertices (or nodes) joined by at most one edge (or link), and no edge going from a vertex to itself. Edges will be called disjoint if they do not share a common vertex. A *matching* in a graph Γ is a set of disjoint edges. A maximal matching is one which can not be enlarged, but these need not all have the same number of edges. A *maximum* matching is one having the largest possible number of edges; this number is called the *matching number* of the graph and will be denoted $\mathrm{mn}(\Gamma)$. A *perfect* or *complete* matching is one which covers every vertex of the graph, i.e., such that every vertex is an end of some edge in the matching. Perfect matchings are necessarily maximum. Computing matching numbers is a basic problem in graph theory but in the case of a tree or forest (a disjoint union of trees) there is a simple algorithm (a trivial case of some more sophisticated ones, cf. e.g. [12] but all we need). Call a vertex at which more than two edges meet a branch point and one met by only one edge an end; a terminal vertex will be one which is either an end or is isolated (not met by any edge). A graph will be called a *chain* if it is connected, has no branch points, and is not an isolated point. Its length is the number of edges. When graphs are directed we can distinguish beginning or "initial" and end or "terminal" vertices but an initial vertex will still be considered terminal in the sense of undirected graphs. A *terminal chain* is a subgraph which is a chain one end of which is a terminal vertex and which has no branch point amongst its interior vertices. (A terminal chain need not be maximal; it can be part of a longer terminal chain.) A terminal edge is one meeting a terminal vertex; it is a terminal chain of length one. (A terminal vertex is frequently called a leaf.)

The matching number of a graph is the sum of those of its components, so to calculate $\mathrm{mn}(\Gamma)$, we may assume that Γ is connected. The *star* of a vertex v, denoted Γ_v, is the subgraph of Γ consisting of all edges meeting v (and their vertices). If $\Gamma = \Gamma_v$ for some v then Γ itself will be called a star. Its matching number is then 1. Suppose now that Γ is not a star. If there is a terminal chain C of length 2 then a maximum matching of $\Gamma \backslash C$ together with a terminal edge of C gives a maximum matching of Γ, so removing C reduces the matching number by exactly 1; similarly if there is an isolated edge. This holds even if Γ has loops. However, if Γ is a tree and there is neither a terminal chain of length 2 nor an isolated edge, then there is a branch vertex v such that at most one edge meeting v is not terminal. Any maximum matching of $\Gamma \setminus \Gamma_v$ can be enlarged to one of Γ by adjoining to it any terminal edge of Γ_v, so removing all edges meeting v also reduces the matching number by exactly one. With this *pruning procedure* one can inductively compute the matching number of a tree, and therefore also of a forest, whose matching number is the sum of those of its trees. Denote the number of vertices of Γ by $\mathrm{vx}(\Gamma)$ and define the *index of a graph* Γ, denoted $\mathrm{idx}(\Gamma)$ to be $\mathrm{vx}(\Gamma) - 2\,\mathrm{mn}(\Gamma)$. Viewing graphs as categories the definition of a product is evident and we conjecture that $\mathrm{idx}(\Gamma_1 \times \Gamma_2) = \mathrm{idx}(\Gamma_1) \cdot \mathrm{idx}(\Gamma_2)$.

The *adjacency matrix* of a graph Γ has rows and columns indexed by the vertices of Γ with 1 in the (i, j) place if vertex i is connected by an edge to

vertex j. This matrix is symmetric; its spectrum, called that of Γ, has been intensively studied. Suppose, however, that Γ is a directed graph. We then define its *skew adjacency matrix* $M(\Gamma)$ to have $+1$ in the (i,j) place if there is an arrow directed from i to j, to have -1 in that place if there is an arrow from j to i, with 0 there otherwise; its rank is denoted $\mathrm{rk}(\Gamma)$. This matrix defines a skew bilinear form B_Γ on the vector space KV spanned over the field K by the vertices V of Γ. Conversely, if we have a skew bilinear form B on a vector space $\mathcal{V}$ then $\mathcal{V}$ is a direct sum of hyperbolic planes (2-dimensional subspaces spanned by elements $v, w \in \mathcal{V}$ with $B(v,w) = 1$) and its radical (those v with $B(v,w) = 0$ for all $w \in \mathcal{V}$). Therefore skew bilinear forms on vector spaces of a fixed dimension are completely determined up to isomorphism by their rank, $\mathrm{rk}(B)$, and with suitable choice of basis are representable by directed graphs Γ or their skew adjacency matrices $M(\Gamma)$.

When Γ is directed we can 'forget' the direction of its arrows to get an undirected graph $|\Gamma|$ but directed graphs Γ and Γ' with $|\Gamma| = |\Gamma'|$ may have different ranks. For example, if Γ is a square with arrows directed cyclically then $\mathrm{rk}(\Gamma) = 2$ but if Γ' is obtained by reversing one arrow then $\mathrm{rk}(\Gamma') = 4$. However, for trees we have the following.

Theorem 1 *If Γ, Γ' are directed trees with $|\Gamma| = |\Gamma'|$ then their skew adjacency matrices are conjugate by a diagonal matrix each diagonal entry of which is ± 1. In particular, B_Γ is isomorphic to $B_{\Gamma'}$ and $\mathrm{rk}(\Gamma) = \mathrm{rk}(\Gamma')$.*

Proof. Suppose that v, v' are vertices of a directed tree Γ with an arrow $v \to v'$. Removing this arrow disconnects the tree. Each vertex which remains connected to v' represents a basis element; changing the direction of the arrow is essentially the same as replacing each of these basis elements by its negative. □

A related concept to matching is that of a *node cover*, i.e., a set T of vertices of Γ such that every edge has an end in T. Here we wish to minimize T. Minimal node covers may have different sizes; one which achieves the absolute minimum is a *minimum cover* and its size, the *cover number*, will be denoted $\mathrm{cn}(\Gamma)$. If M is a matching and T a node cover, then no vertex in T can cover more than one edge in M. Therefore $\#M \leq \#T$, so $\mathrm{mn}(\Gamma) \leq \mathrm{cn}(\Gamma)$. In general this inequality is strict. (The triangle has vertex number equal to 2 but matching number 1.) However, for a *bipartite* graph, i.e., one in which the vertices can be partitioned into two disjoint sets with every edge connecting a vertex in one to a vertex in the other, Menger's Theorem asserts that equality holds. One may think of a bipartite graph as having its vertices colored, say either black or white, with edges always connecting vertices of different colors. Trees are bipartite.

5 The graph $\Gamma(S)$

Suppose that $\mathfrak{g}$ is a Lie subalgebra of $\mathrm{sl}(n)$ which contains the Cartan subalgebra $\mathfrak{h}$ of traceless diagonal matrices. It is then spanned by these matrices together

with all $e_{ij} \in \mathfrak{g}$. Let $\Pi(\mathfrak{g})$ denote the support of $\mathfrak{g}$, i.e., the set of those pairs of indices (i,j) for which $e_{ij} \in \mathfrak{g}$, together with the diagonal pairs (i,i); for $\mathfrak{g} = \mathcal{P}(n,m)$ we will write simply $\Pi(n,m)$. Given a small functional F_S on $\mathfrak{g}$ (where tacitly $S \subset \Pi(\mathfrak{g})$) we now define a directed graph $\Gamma(S)$ with vertices the union of (i) the set of those e_{ij} in $\mathfrak{g}$ where (i,j) is neither in S nor on the diagonal and (ii) a vertex labeled d_s for each $s \in S$. Writing B_S for B_{F_S} draw an arrow $e_{ij} \to e_{kl}$ whenever $B_S(e_{ij}, e_{kl}) = 1$, i.e., if $j = k$ and $(i,l) \in S$, and for every $s \in S$ add an arrow $d_s \to e_s$ since we also have $B_S(d_s, e_s) = 1$.

The decomposition of $\mathfrak{g}$ (which here need not be Frobenius) into eigenspaces of $\mathrm{ad}(D_S)$ also decomposes $\Gamma(S)$ into disjoint subgraphs, one for each pair of eigenvalues $(m, 1-m)$, since an e_{ij} in the eigenspace for the eigenvalue m can only be linked to one for the eigenvalue $1-m$ while d_s can only be linked to e_s (the eigenvalues for which are 0 and 1, respectively). The eigenspace components of $\Gamma(S)$ may themselves decompose further. Each component and hence all of $\Gamma(S)$ is therefore bipartite. When $\mathfrak{g}$ is Frobenius one can in principle use $\Gamma(S)$ to calculate the associated solutions to the CYBE, and we do so in several cases. The graph $\Gamma(S)$ then effectively organizes the inversion of B_S, which may be large but sparse. With the preceding notations we have the following.

Theorem 2 *If $\Gamma = \Gamma(S)$ is a tree then* $\mathrm{rk}(\Gamma) = \mathrm{rk}(B_S) = 2\,\mathrm{mn}(|\Gamma|)$. *If moreover F_S is regular then* $\mathrm{idx}(\Gamma_S) = \mathrm{idx}(\mathfrak{g})$.

Proof. Note that while Γ is a directed graph, by the remarks in the preceding section its rank does not depend on the directions of the arrows so it must depend only on $|\Gamma|$, so when there can be no confusion we will write simply Γ for $|\Gamma|$. The theorem is obvious for Γ a star. (Taking the vertex at the center of the star as first basis element the matrix of B has non-zero elements only in the first row and first column.) For larger Γ suppose that the theorem holds for all trees with fewer vertices than Γ and apply the pruning procedure of the previous section. If there is a terminal chain C of length 2 then by hypothesis the theorem holds for $\Gamma \setminus C$, the matching number of which is one less than that of Γ. Take the terminal vertex which has been removed and that connected to it (also removed) as the first and second basis elements. Since only the second is linked to any elements of $\Gamma \setminus C$ it is evident that the matrix corresponding to $\Gamma \setminus C$, which is obtained from that of Γ by removing the first and second rows and columns, has rank exactly two less than that of Γ. When the pruning removes a terminal star Γ_v the argument is a slight elaboration of the preceding. Take v as the first vertex, followed by the others of the star and then those of $\Gamma \setminus \Gamma_v$. Since the only vertex of Γ_v linked to $\Gamma \setminus \Gamma_v$ is v, removing the first row and column of the full matrix of the form reduces it to the direct sum of a zero matrix and that corresponding to $\Gamma \setminus \Gamma_v$. It is evident then that the difference in ranks is again exactly two, regardless of the number of vertices in Γ_v. This proves the first assertion of the theorem; the second follows. □

The graph $\gamma(S)$ is always a tree and we should like to be able to show that this implies the same for the components of $\Gamma(S)$ (which are, in any case,

bipartite), but that for the moment is an open question. The case of most interest for us is, however, that where $\gamma(S)$ is a rooted tree (all arrows being directed away from the root).

Theorem 3 *If $\gamma(S)$ is a rooted tree then so are all components of $\Gamma(S)$.*

PROOF. Observe first that if $s \in S$ then the only arrow that can terminate on e_s is $d_s \to e_s$, so in the proof we may disregard all d_s (which will be the root of any component in which it appears) and consider only vertices e_{ij} of $\Gamma(S)$ with $i \neq j$. Suppose, if possible, that we had arrows $\varepsilon_{ij} \to e_{jk} \leftarrow e_{hj}$ in $\Gamma(S)$. Then in $\gamma(S)$ we must have $i \to k \leftarrow h$, contradicting the assumption that $\gamma(S)$ is a rooted tree with all arrows directed away from the root.□

6 Isotropic and Lagrangian subspaces

Let V denote a vector space over a field of characteristic different from 2 endowed with a skew bilinear form $B(-,-)$. A subspace W is called isotropic if $B(w_1, w_2) = 0$ for all $w_1, w_2 \in W$; equivalently it is contained in its orthogonal complement $W^\perp = \{w' \in V | B(w, w') = 0, \text{all } w \in W\}$. If B is non-degenerate then $\dim V = 2\ell$ is even and every maximal isotropic subspace L has dimension ℓ; such subspaces are called Lagrangian. One has $V = L \oplus L^\perp$, where $L^\perp$ is again Lagrangian and choosing any basis $\{v_1, \ldots, v_\ell\}$ in V there is a dual basis $\{v'_1, \ldots, v'_\ell\}$ with $B(v_i, v'_j) = \delta_{ij}$. The matrix of B relative to the basis $\{v_1, \ldots, v_\ell, v'_1, \ldots, v'_\ell\}$ then has the form $\begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}$ the inverse of which is just its negative.

Suppose that we have a Frobenius Lie algebra $\mathfrak{f}$ with Frobenius functional F and form $B = B_F$. The computation of the associated r-matrix becomes trivial if we can write $\mathfrak{f}$ as a direct sum of a pair of Lagrangian subspaces $\mathfrak{f} = L \oplus L'$ with an explicit duality between them. For with a basis of $\mathfrak{f}$ consisting of a basis x_i of L followed by the dual basis x'_i of L' the matrix of B has the form $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, where I is an identity matrix of size the dimension of L. The inverse of this matrix is just its negative, and since we could multiply by any non-zero scalar, we can simply take $r = \sum x_i \wedge x'_i$.

Theorem 4 *A Frobenius Lie algebra $\mathfrak{f}$ whose principal element has only integral eigenvalues can be decomposed into a direct sum of Lagrangian subspaces in duality with each other, one of which is a Lie subalgebra and the second a module over the first under the Lie multiplication. In particular this holds for Frobenius subalgebras of $\mathrm{sl}(n)$ which contain the Cartan subalgebra.*

PROOF. It follows from [6] (or explicit trivial calculation) that principal elements $\hat{F}$ have adjoints $\operatorname{ad}\hat{F}$ whose eigenvalues are integers. Letting $\mathfrak{f}_m$ denote the eigenspace for the integer m, the dual of which is $\mathfrak{f}_{1-m}$, one see that

$\mathfrak{f}_{\text{even}} = \sum_{m\,\text{even}} \mathfrak{f}_m$ and $\mathfrak{f}_{\text{odd}} = \sum_{m\,\text{odd}} \mathfrak{f}_m$ are dual Lagrangian subspaces. Since $[\mathfrak{f}_m, \mathfrak{f}_n] \subset \mathfrak{f}_{m+n}$, $\mathfrak{f}_{\text{even}}$ is a Lie subalgebra and $\mathfrak{f}_{\text{odd}}$ is a module over $\mathfrak{f}_{\text{even}}$ under the multiplication in $\mathfrak{f}$. $\square$

Suppose that one can readily find only one Lagrangian subspace L (not necessarily a subalgebra) with a complement L' which is not necessarily Lagrangian. If a basis $\{v_1, \dots, v_\ell\}$ of L is given then we can still find in L' a dual basis $\{v'_1, \dots, v'_\ell\}$ but now the matrix of B relative to the basis $\{v_1, \dots, v_\ell, v'_1, \dots, v'_\ell\}$ will have the form $M = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & Q \end{pmatrix}$ for some $\ell \times \ell$ matrix Q. Then $M^{-1} = \begin{pmatrix} Q & -I_\ell \\ I_\ell & 0 \end{pmatrix}$ so no computation is required to find the associated r-matrix beyond finding the dual basis to $\{v_1, \dots, v_\ell\}$.

7 The cyclic functional

The *cyclic functional* on $\mathcal{P}(n, m)$ (with $(n, m) = 1$) is the one in which we are most interested. Its associated tree graph will be denoted simply by $\gamma = \gamma(n, m)$. To define it, write the integers $1, \dots n$ in the order $1, m+1, 2m+1, \dots, (n-1)m+1, 1$ where all $km+1$ are understood to be reduced modulo m except n itself, and the order is understood to be cyclic. For example, with $n = 13, m = 5$ one has $\dots 1, 6, 11, 3, 8, 13, 5, 10, 2, 7, 12, 4, 9, 1, 6, 11, \dots$. As in [7], divide the array into *strings* of consecutive entries inside which the integers are increasing in their natural order. In the example the strings are $(1, 6, 11), (3, 8, 13), (5, 10), (2, 7, 12), (4, 9)$; this will be called the first *cycle of strings*. The directed graph γ as before has vertices the integers $1, \dots, n$ to which we add directed edges beginning with those joining successive integers in each string. In the illustration we add $1 \to 6 \to 11$, $3 \to 8 \to 13$, $5 \to 10$, $2 \to 7 \to 12$. Now consider the "drops" in the original cyclic order, i.e., those integers which are less than their predecessors (the first integers of the strings); here they are $3, 5, 2, 4, 1$. Arrange these similarly in ascending strings, $(1, 3, 5), (2, 4)$ (note the cyclic order), giving the second cycle of strings. Now draw arrows from the *larger* to the *smaller* integers in the strings: $1 \leftarrow 3 \leftarrow 5$, $2 \leftarrow 4$. Next, instead of removing the drops, remove the "rises", those numbers which, reading in *reverse* cyclic order, are larger than their predecessors (the last elements in the strings). Here one has only $5, 4$. Now go back to the first procedure, grouping them into ascending strings. In this case, we have only one arrow to add, $4 \to 5$. (Remember that the cyclic order is $4, 5, 4$.) In general we continue alternating between drawing arrows from smaller integers to larger ones and from larger integers to smaller ones until the process ends with but a single group. This also defines the set $S = S_{n,m}$ supporting the cyclic functional, denoted $F_{n,m}$; it is the set of those pairs (i, j) for which there is an arrow from i to j in γ. Isomorphic algebras, in particular $\mathcal{P}(n, m)$ and $\mathcal{P}(n, n-m)$, generally do not have isomorphic trees. The simplest example of this is $\mathcal{P}(5, 2)$, whose tree has root 2, at which it branches, and $\mathcal{P}(5, 3)$, whose tree has root 3 but there is

no branching there. Note that the mth superdiagonal is filled by S, i.e., all $(i, i+m)$ in $\Pi(n,m)$ are in S;

Theorem 5 *The graph $\gamma(n,m)$ is a rooted tree with all branches directed away from the root; with the exception of the unique root, every $j \in \{1,\dots,n\}$ has a unique immediate predecessor i such that $i \to j$.*

PROOF. An integer will appear in a cycle of strings only if it has no predecessor in any preceding cycle, so for any integer there can be at most one predecessor. Since the process continues until the cycle consists of just a single string, there can only be one integer in $\{1,\dots,n\}$ without a predecessor, and that will be the root. □

We will say more generally that a vertex v of a directed graph has another vertex w as a predecessor if there is an arrow $w \to v$. Viewing S for the moment as an $n \times n$ matrix, the theorem shows that there is precisely one non-zero element (equal to 1) in each column except for that column corresponding to the root (which in the example is column 4). It follows for $\Gamma(n,m)$ that predecessors, when they exist, are unique, so each component of $\Gamma(n,m)$ is therefore itself a rooted tree. This is an essential feature of the cyclic functional.

The directed tree $\gamma = \gamma(n,m)$ of the cyclic functional can be built recursively. If $n > 2m$ then $\gamma(n-m,m)$ is a subgraph of $\gamma(n,m)$ and the latter is obtained from the former by attaching an outgoing arrow to every vertex i with $n-2m+1 \le i \le n-m$ with the integer $i+m$ at the point of the arrow. Note that $n-m+1,\dots,m$ are the ends of $\gamma(n,m)$ (vertices with no outgoing arrow) so one may view the reduction as the removal of all ends together with the unique arrows terminating on them. In this case, which we will call *stable reduction* the root of $\gamma(n-m,m)$ coincides with that of $\gamma(n,m)$. If $n-m<m$ then stripping all the $n-m$ ends (whose labels are $m+1,\dots,n$) from $\gamma(n,m)$ produces a graph which as a directed tree but without labeled vertices is identical with $\gamma(m,2m-n)$. To get the correct labeling one must now replace every i with $m+1-i$. It follows that if $\rho(n,m)$ is the root of $\gamma(n,m)$ and $\rho(m,2m-n)$ that of $\gamma(m,2m-n)$ then $\rho(n,m) = m+1-\rho(m,2m-n)$. Since the numbering and root both change, we call this case *unstable reduction*. For example, $\gamma(17,6)$ reduces stably to $\gamma(11,6)$, so the root does not change, but $\gamma(11,6)$ reduces unstably to $\gamma(6,1)$. The root of the latter is 1 (and that of any $\gamma(n,1)$ is clearly 1), so the root of $\gamma(11,6)$ is $(6+1)-1=6$, which therefore is also the root of $\gamma(17,6)$.

Theorem 6 *The cyclic functional is Frobenius.*

PROOF. We show that $F_{n,m}$ is Frobenius if and only if the same is true for $F_{n',m'}$, where $(n',m') = (n-m,m)$ if $n > 2m$ (stable case) or $(n',m') = (m,2m-n)$ if $n < 2m$ (unstable case); ultimately $(n,m) = (2,1)$ where the theorem is obvious. The reduction processes are similar in that in each case we remove two disjoint blocks with the same numbers of elements from $\Pi(n,m)$. In the stable case, the first block to be removed consists of the last m rows of

$\Pi(n,m)$; it is a block of m rows and $n-m$ columns. After that the second block to be removed consists of the last m columns of what remains; it is a block of $n-m$ rows and m columns. Note that the first block contains no element of S, while the second contains one element of S in each column. The resulting configuration is that of $\Pi(n-m,m)$ in its standard position. In removing the second block we removed m elements of S. Those that remain are in the correct positions for $S_{n-m,m}$. In the unstable case, the first block to be removed consists of those (i,j) with $n-m+1 \le i \le n$, $m=1 \le j \le n$; it is a block of m rows and $n-m$ columns. After that the second block removed consists of those (i,j) with $1 \le i \le n-m, n-m+1 \le j \le n$; it is a block of $n-m$ rows and m columns. Again, the first block contains no element of S but now the second block contains one element of S in each row. The configuration that remains in the unstable case is not that of any usual $\Pi(n',m')$, but after rotation through a half circle becomes that of $\Pi(m,n-m)$. Since there are now m rows and columns, this amounts to replacing (i,j) in the reduced configuration by $(m+1-i,m+1-j)$. In removing the second block we removed $n-m$ elements of S. After the rotation, they are in the correct position for $S(m,2m-n)$. Note that in both cases the two blocks have the same number of elements, the first block contains no elements of S, the second block contains no elements on the diagonal, and that the number of diagonal elements of the first block is equal to the number of elements of S in the second.

These steps are illustrated in the following figure, where the first block is marked by the solid black squares ■ and the second by the open ones □. Entries in S are marked by **x**, including those inside the open squares. (There can be none in the locations marked by the black squares.) Entries outside $\Pi(7,3)$ must be zero and are marked by dots.

$$\begin{pmatrix} * & * & * & \mathbf{x} & \square & \square & \square \\ * & * & \mathbf{x} & * & \boxtimes & \square & \square \\ \mathbf{x} & * & * & * & \square & \boxtimes & \square \\ \cdot & \cdot & \cdot & * & \square & \square & \boxtimes \\ \cdot & \cdot & \cdot & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \cdot & \cdot & \cdot & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \cdot & \cdot & \cdot & \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{pmatrix} \rightsquigarrow \begin{pmatrix} * & \square & \square & \boxtimes \\ * & * & \mathbf{x} & \blacksquare \\ \mathbf{x} & * & * & \blacksquare \\ \cdot & \cdot & \cdot & \blacksquare \end{pmatrix} \rightsquigarrow \begin{pmatrix} * & \cdot & \cdot \\ * & * & \mathbf{x} \\ \mathbf{x} & * & * \end{pmatrix} \qquad (3)$$

FIGURE 1. `Steps in the reduction of` $\Pi(7,3)$: `Stable reduction to` $\Pi(4,3)$, `followed by unstable reduction to` $\Pi(3,2)$. `The last matrix must be rotated through` 180° `to bring` S `into the standard position for` $\gamma(3,2)$.

Consider next what happens to the graph $\Gamma(S)$ in the reduction process. The first (black square) block removed contains no element of S, so none of its entries has a successor (an element of which it is the predecessor), but every element of the first block which is not on the diagonal has a (necessarily unique) predecessor in the second (white square) block, and that predecessor can not be an element of S. This defines a bijective map from the non-diagonal elements of the first block to those elements of the second block which are not in S.

An element of the second block may have no predecessor, but if it does, that predecessor (which may be an element of S) is not contained in the second block and therefore remains after the reduction. Letting $\Gamma' = \Gamma(S')$ denote the graph of what remains after the reduction, to build $\Gamma = \Gamma(S)$ from Γ' one must do the following: First, if (k,l) is an element of the first (black square) block whose predecessor (j,k) in the second (white square) block itself has a predecessor (i,j) then one must attach a terminal chain of length 2: $e_{ij} \to e_{jk} \to e_{kl}$. (Here e_{ij} is necessarily a vertex of $\Gamma(S')$.) If (k,l) is an element of the first block whose predecessor (j,k) in the second block has no predecessor then adjoin a disconnected arrow $e_{jk} \to e_{kl}$. Finally, for each $s \in S \setminus S'$ adjoin another disconnected arrow from a new vertex labeled d_s (the diagonal element dual to e_s) to e_s. Note that in every case when an arrow points from a vertex v to its successor v' one has $F_S([e_v, e_{v'}]) = 1$, and conversely.

It is evident now that the matching number of Γ is the matching number of Γ' plus half the number of elements in the two blocks removed (or the number in either single one of them) since to every pair of elements (non-diagonal element of first block, its predecessor) or (diagonal element of first block, element of S in its column) we have either attached to some element of Γ' a terminal chain of length 2 or have adjoined a disjoint link. The reduction brings us to some smaller $\Pi(n', m')$ (where again $(n', m') = 1$) with its corresponding cyclic form; if the latter is Frobenius then so is that with which we started. Since the smallest case is evident on inspection, this ends the proof. □

In the proof we have, in effect, recursively constructed the Γ there from Γ'. We conclude this section by recursively constructing $\Gamma(7,3)$ starting with $\Gamma(3,2)$: $\{e_{12} \to e_{23}, d_{13} \to e_{13}, d_{21} \to e_{21}\}$; here $d_{13} = \varepsilon_1 + \varepsilon_2$ and $d_{21} = \varepsilon_2$. Since the reduction from $\Gamma(4,3)$ to $\Gamma(3,2)$ was unstable we must now complement all indices with respect to 4 (d_{13} becomes $d_{31} = \varepsilon_3 + \varepsilon_2$ and d_{21} becomes $d_{23} = \varepsilon_2$) and then adjoin the new arrows. To indicate which arrows are new and which have come from (the rotated) $\Gamma(3,2)$ we will indicate the latter by a double arrow $\Rightarrow$ and the new arrows by a single arrow $\to$.

$$e_{32} \Rightarrow e_{21} \to e_{13} \to e_{34} \quad d_{23} \Rightarrow e_{23} \quad d_{31} \Rightarrow e_{31} \quad d_{14} \to e_{14} \quad e_{12} \to e_{24}$$

FIGURE 2. The graph $\Gamma(4,3)$

Note that d_{31} is still $e_3 + \varepsilon_2$ and d_{23} remains ε_2, while $d_{14} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$. There are 12 vertices (the dimension of $\mathcal{P}(4,3)$) and the matching number of this graph is 6. The graph below of $\Gamma(7,3)$ is arranged so that arrows stemming from $\Gamma(3,4)$ are vertical and new ones, denoted by $\dashrightarrow$, are horizontal. There are 11 components, 36 vertices (dimension of $\mathcal{P}(7,3)$), and the matching number

can be seen to be 18.

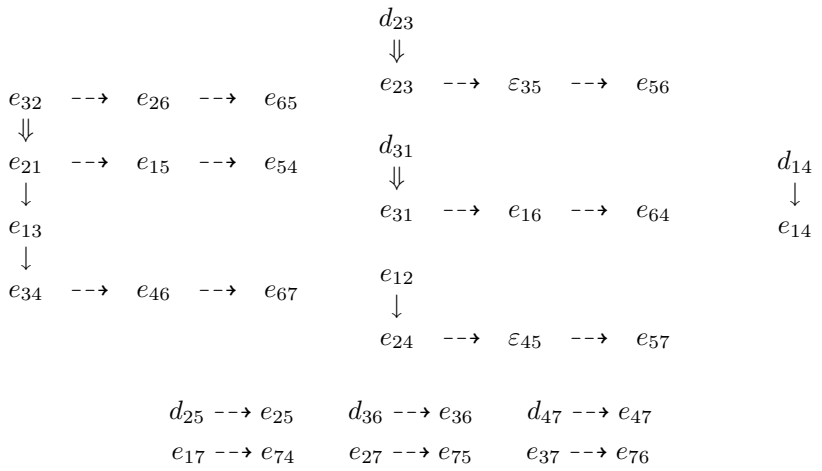


FIGURE 3. The graph $\Gamma(7,3)$

The eigenvalue pair $(m, 1-m)$ to which each component of this graph belongs is determined by its root; the values are easily obtained by using the principal element $D_{S_{\text{cyclic}}}$ of equation (2). (For example, the root e_{32} of the largest component belongs to the eigenspace with $m = -1$.) The $\Gamma(S)$ here has a unique perfect matching; this will always be the case for any small Frobenius functional F_S.

8 Solutions to the CYBE from graphs

The solution to the CYBE derived from $\mathcal{P}(n,m)$ using the cyclic functional will be denoted $r(n,m)$. It is naturally a sum of terms corresponding to the components of $\Gamma(n,m)$. The bilinear form defined by the functional will for the moment be denoted simply B.

8.1 $\mathcal{P}(4,3)$ and $\mathcal{P}(7,3)$

The graph $\Gamma(4,3)$ (Figure 2) has five components of which the last four are just isolated links. The four corresponding summands of the solution to the CYBE are $d_{23} \wedge e_{23},\ d_{31} \wedge e_{31},\ d_{14} \wedge e_{14},\ e_{12} \wedge e_{24}$. To compute the summand for the first component, which illustrates the more general situation, notice that a terminal chain of length 2 has been added to a component of the graph $\Gamma(3,2)$; the reduction process of the preceding section shows that any $\Gamma(n,m)$ (where $(n,m) = 1$) is built from a smaller one by the addition of isolated links and terminal chains of length 2. The graph encodes that $B(e_{13}, e_{34}) = 1$ and $B(e_{21}, e_{13}) = 1$, so $B(e_{21} + e_{34}, e_{13}) = 0$. Since $B(e_{21}, e_{34}) = 0$ we also have $B(e_{32}, e_{21} + e_{34}) = 1$. We can therefore separate the first component into two

components, $e_{32} \Rightarrow e_{21} + e_{34}$ and $e_{13} \rightarrow e_{34}$. By replacing e_{21} (to which the terminal chain was attached) by $e_{21}+e_{34}$ we have in effect detached the terminal chain (which is now reduced to an isolated link) from the previous graph $\Gamma(3,2)$. What remains is now also just an isolated link so the more general process of detaching terminal chains of length two by which the larger graph has been built from the smaller is ended. The summand corresponding to the first component of $\Gamma(3,4)$ is thus $e_{32} \wedge (e_{21} + e_{34}) + e_{13} \wedge e_{34}$. Inserting the values for the various d_s we therefore have

$$\begin{aligned} r(4,3) \ &= e_{32} \wedge (e_{21} + e_{34}) + e_{13} \wedge e_{34} + \\ &\qquad \varepsilon_2 \wedge e_{23} + (\varepsilon_2 + \varepsilon_3) \wedge e_{31} + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \wedge e_{14} + e_{12} \wedge e_{24}. \end{aligned}$$

Analyzing the first component of $\Gamma(7,3)$ in the same way, disconnecting the three terminal chains (dotted arrows) that have been added to a component of $\Gamma(4,3)$ reduces it to

$$\begin{gathered} (e_{32} + e_{65}) \Rightarrow (e_{21} + e_{54}) \rightarrow e_{13} \rightarrow (e_{34} + e_{67}) \\ e_{26} \dashrightarrow e_{65} \qquad e_{15} \dashrightarrow e_{54} \qquad e_{46} \dashrightarrow e_{67} \end{gathered}$$

We can now treat the first line above exactly as before, so the contribution of the first component of $\Gamma(7,3)$ to $r(7,3)$ is

$$(e_{32}+e_{65})\wedge[(e_{21}+e_{54})+(e_{34}+e_{67})]+e_{13}\wedge(e_{34}+e_{67})+e_{26}\wedge e_{65}+e_{15}\wedge e_{54}+e_{46}\wedge e_{67}$$

It is not difficult now to compute the 13 additional summands of $r(7,3)$; we omit it. (The isolated links contribute 7 and the remaining three components each contribute two.)

8.2 $\mathcal{P}(n,1)$

The cyclic functional F for $\mathcal{P}(n,1)$ reduces to what is sometimes called the *prime functional*. Its carrier is $S = \{(1,2),(2,3),\ldots,(n-1,n)\}$. One has

$$\begin{aligned} &\gamma(S) = 1 \rightarrow 2 \rightarrow \cdots \rightarrow (n-1) \rightarrow n, \quad d_{(i,i+1)} = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i, \\ &D_S = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + \varepsilon_{n-1} \\ &\quad = (1/2)[(n-1)e_{11} + (n-3)e_{22} + \cdots + (3-n)e_{n-1,n-1} + (1-n)e_{nn}]. \end{aligned}$$

The eigenspace of $\mathrm{ad}(D_S)$ for the eigenvalue m is the Cartan subalgebra when $m = 0$ and otherwise the mth superdiagonal (subdiagonal if m is negative). A closed form for the associated solution to the CYBE was given in [7]. Using the graph $\Gamma = \Gamma(n,1)$ we see here why it has the given form. In this simple case Γ

is a disjoint union of chains.

$$
\begin{gathered}
d_{i,i+1} \to e_{i,i+1}, \qquad i = 1, \ldots, n-1 \\
(1,3) \to (3,2) \to (2,4) \to (4,3) \to \cdots \to (n-2,n) \to (n,n-1) \\
(1,4) \to (4,2) \to (2,5) \to (5,3) \to \cdots \to (n-3,n) \to (n,n-2) \\
\cdots \\
(1,n-1) \to (n-1,2) \to (2,n) \to (n,3) \\
(1,n) \to (n,2)
\end{gathered}
$$

These are, respectively, the components of Γ for the eigenvalue pairs $(0,1)$ (the $n-1$ short chains of the first row) $(2,-1),(3,-2),\ldots,(n-2,3-n),(n-1,2-n)$. Applying the procedure described in the preceding section, from the last chain one has $e'_{1n} = e_{n2}$, from the next that $e'_{2n} = e_{n3}$ and $e'_{1,n-1} = e_{n-1,2} + e_{n,3}$, and so forth. From the first chain we get, in particular, that $e'_{1,3} = e_{3,2} + e_{4,3} + \cdots + e_{n,n-1}$. Collecting terms $\sum_{i<j} e_{ij} \wedge e'_{ij}$ and noting that $d_{i,i+1} \to e_{i,i+1}$ (so $d'_{i,i+1} = e_{i,i+1}$) we get the closed form of [7] for the solution to the CYBE associated to $\mathcal{P}(n,1)$. As given there, with terms collected in slightly different order and writing simply d_p for $d_{p,p+1}$ it is

$$
r(n,1) \quad = \quad \sum_{p=1}^{n-1} d_p \wedge e_{p,p+1} + \sum_{i<j} \sum_{m=1}^{j-i-1} e_{i,j-m+1} \wedge e_{j,i+m}.
$$

The carrier S of the prime functional is unchanged by reflection across the antidiagonal, so the prime functional will work also for $\mathcal{P}(n,n-1)$. However, $m=1$ and $m=n-1$ are the only cases where the prime functional is Frobenius; in other cases, while the eigenspace for $n-1$ always has dimension 1 that for $2-n$ (which should be its dual) vanishes.

8.3 $\mathcal{P}(n,2)$

Note that n must be odd. Here $\Gamma(n,2)$ again consists only of chains. One has

$$
\begin{gathered}
S = \{(2,1),(1,3),(2,4),(3,5),\ldots,(n-2,n)\}, \\
\gamma(2,n) \quad = \quad (n-2) \leftarrow (n-4) \leftarrow \cdots \leftarrow 4 \leftarrow 2 \to 1 \to 3 \to 5 \cdots \to n.
\end{gathered}
$$

From this one finds that

$$
\begin{aligned}
d_{21} &= \varepsilon_2 + \varepsilon_4 + \cdots + \varepsilon_{n-2}, \\
d_{2k+2,\,2k+3} &= -(\varepsilon_{2k+2} + \varepsilon_{2k+4} + \cdots + \varepsilon_{n-3} + \varepsilon_{n-1}), \\
d_{2k+1,\,2k+3} &= \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \cdots + e_{2k+1} - d_{2k+2,2k+3} \\
&= (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \cdots + e_{2k+1}) + (\varepsilon_{2k+2} + \varepsilon_{2k+4} + \cdots + \varepsilon_{n-3} + \varepsilon_{n-1})
\end{aligned}
$$

For $\Gamma(n,2)$, $n \geq 5$ one has

$$d_s \to e_s, \quad \text{all} \quad s \neq (2,1)$$
$$e_{12} \to e_{23} \to e_{34} \to \cdots \to e_{n-1,n}$$
$$d_{21} \to e_{21} \to e_{14} \to e_{43} \to \cdots \to e_{n-4,n-1} \to e_{n-1,n-2}$$
$$\text{and for all } 5 \leq j \leq n,\ k \geq 0,$$
$$e_{1,j} \to e_{j,3} \to e_{3,j+2} \to e_{j+2,5} \to \cdots \to e_{2k+1,2k+j} \to e_{2k+j,2k+3} \to \cdots$$
$$e_{2,j} \to e_{j,4} \to e_{4,j+2} \to e_{j+2,6} \to \cdots \to e_{2k+2,2k+j} \to e_{2k+j,2k+4} \to \cdots$$

where a chain terminates when any index exceeds n. It follows that

$$r(1,2) = \sum_{s\in S, s\neq(2,1)} d_s \wedge e_s + d_{21} \wedge (e_{21} + \sum_{k\geq 2} e_{2k,2k-1}) + \sum_{i<j, j\neq i+2} e_{ij} \wedge \sum_{k\geq 0} e_{j+2k,i+2k+2}$$

where the sums terminate when the indices are out of range. The principal element is

$$\hat{F} = \mathrm{diag}(0,1,-1,0,-2,-1,-3,\ldots,(5-n)/2,(1-n)/2) + [(n-1)(n-3)/4n]\,\mathbf{I_n}.$$

9 Some other small Frobenius functionals

There are many small Frobenius functionals that can be defined for $\mathcal{P}(n,m)$. We give several in this section just to show the variety but consider the Dergachev-Kirillov functional separately.

9.1 The subprime functional

For $m > 1$ the simplest functional after the prime is the *subprime*, defined by taking S to be the union of the sets $(i, i+m), i = 1,\ldots,n-m$ and $(i+1,i), i = 1,\ldots,m-1$. No column contains more than one element of S and the only one without one is the mth. For $n \equiv -1 \mod m$ this is just the cyclic functional. The subprime functional is still Frobenius when $n \equiv 1 \mod m$; the proof is by a reduction process almost identical with that in the cyclic case and the associated r matrix is still relatively simple to construct, at least for small n. However it is only for $n \equiv \pm 1 \mod m$ that the subprime functional is Frobenius.

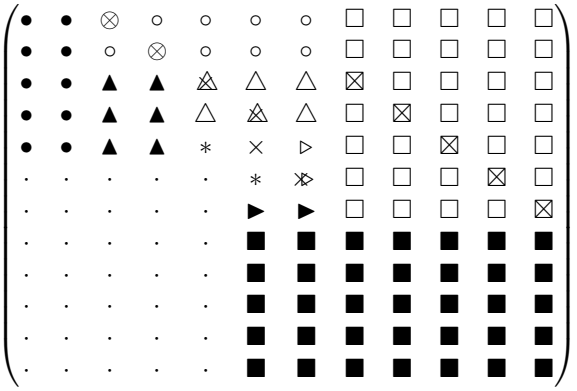
9.2 The upper triangular functional

The *upper triangular* S is a modification of the cyclic functional with the property that S is contained entirely above the diagonal. As with the cyclic functional, first write the integers $1,2,\ldots,n$ in the order $1, m+1, 2m+1, \ldots$ where the entries are understood modulo m except for m itself and the order is cyclic (so we end again with 1). For example, with $n = 12, m = 5$ we

have $1, 6, 11, 4, 9, 2, 7, 12, 5, 10, 3, 8, 1$ where we have repeated the first integer to emphasize the cyclic order. If $n > 2m$ read the sequence forwards, remove the m largest numbers from the sequence, and if j is one of these and i its predecessor in the sequence, take the pair (i, j) into S. View what remains of the sequence as associated to $\mathcal{P}(n', m')$ with $n' = n - m, m' = m$. However, if $n < 2m$ remove the $n - m$ smallest numbers and if j' is one of these and i' its predecessor take (j', i') (note the transposition) into S. Think of reading the sequence backwards. Now view what remains of the sequence as associated to $\mathcal{P}(n', m')$ with $n' = m, m' = n - m$ *except that the indexing has been shifted up by* $n - m$ since $1, \ldots, n - m$ have been removed. In either case, if now $n' > 2m'$ continue reading the sequence in the same direction in which it was last read, otherwise reverse direction. So for $\mathcal{P}(12, 5)$ we first strike $8, 9, 10, 11, 12$, taking $(3, 8), (4, 9), (5, 10), (6, 11), (7, 12)$ into S and leaving the sequence $1, 6, 4, 2, 7, 5, 3, 1$. Now we proceed as if we were in the case $n = 7, m = 5$, so we must strike the the two smallest integers, 1 and 2, whose predecessors are respectively 3 (remember the cyclic order!) and 2, so we take $(1, 3), (2, 4)$ into S. The remaining sequence is $6, 4, 7, 5, 3, 6$ and we proceed as if it were the case of $n = 5, m = 2$. *We now continue reading the sequence in the same direction as before*, namely backwards, removing the smallest entries 3 and 4, and taking $(3, 5), (4, 6)$ into S. This leaves $6, 7, 5, 6$ with $n = 3, m = 2$. We must reverse direction again, and since we are again reading in the *forward* direction we strike only the one *largest* integer, 7, and take $(6, 7)$ into S. The sequence has been reduced to $6, 5, (6)$ and we are in final case of $n = 2, m = 1$. We do not reverse direction and take $(5, 6)$ into S, producing an upper triangular S for which $\gamma(S)$ is obviously a tree. Note that we started with $m = 5$ and lastly took in $(5, 6)$; the final entry into S will always be $(m, m + 1)$.

The foregoing is actually a prescription for a reduction procedure removing pairs of blocks analogous to that in the cyclic case; here, too, it gives an inductive proof that S is Frobenius. This is illustrated in the following diagram for $\mathcal{P}(12, 5)$ where dots indicate the places that must be filled with zeros and positions which have some manner of "x" in them are those in S. The first entries taken into S were (3,8), (4,9), (5,10), (6,11), and (7,12); these are in the positions indicated by $\boxtimes$. The last five rows are removed, which removes the 5×10 block indicated by the black squares $\blacksquare$ and then the last 5 columns, which removes the 10×5 block indicated by the squares $\square$ (including those in the positions already marked by $\boxtimes$). What remains is the diagram for $\mathcal{P}(7, 5)$, where now $n < 2m$. If this were transposed to get the case of $\mathcal{P}(7, 2)$ then the previous procedure would take the positions indicated by $\otimes$ into S, namely (1,3) and (2,4). Now one must remove the first two columns, removing the 5×2 block indicated by the "bullets" $\bullet$, and then the first two rows, which will remove the 2×5 block indicated by the small circles $\circ$. One is now in the $\mathcal{P}(5, 2)$ case but the remaining columns and rows begin with the third, so (3,5) and (4,6) are taken into S; these are in the positions indicated by $\boxtimes$ (an approximation to an x included in a triangle). Now the first two of the remaining columns are removed (columns 3 and 4 of the original), which removes the 3×2 block indicated by the black triangles $\blacktriangle$, and then first two of the remaining rows are

removed (rows 3 and 4 of the original), which removes the 2×3 block indicated by the triangles $\triangle$ (including those already marked $\blacktriangle$. What is left is a $\mathcal{P}(3,1)$ where (5,7), in the position marked by an x in a right pointing triangle$\rtimes$ is taken into S, and the block marked by the black right pointing triangles $\blacktriangleright$ and the open right pointing triangles $\triangleright$ are removed. This leaves finally a $\mathcal{P}(2,1)$ and the last entry taken into S, namely (5,6), is marked simply $\times$.



The proof that the upper triangular S is Frobenius follows the same reduction procedure as for the cyclic S. The blocks have been removed in the example as they would be in the reverse induction. (That predecessors now need not be unique causes a only a slight complication in the proof; there are natural choices.) To compute the r matrix one shows that the subspace L spanned by all $e_{ij}, i \neq j$ in the first block removed in every pair (the black ones in the illustration) together with all $e_s, s \in S$ is Lagrangian. Its dual L' consists of the Cartan subalgebra of $\mathrm{sl}(n)$ together with all $e_{ij}, (i,j) \notin S$ in the second block removed in every pair. Although L' is generally not Lagrangian the computation is not difficult, cf. Section 6.

10 Seaweed algebras

Dergachev and Kirillov [4] define a *seaweed* subalgebra $\mathfrak{k}$ of a simple Lie algebra $\mathfrak{g}$ to be one generated by a Cartan subalgebra together with the root spaces of some subset of the simple roots, both positive and negative. (The suggestive name comes from the picture of such an algebra when $\mathfrak{g} = \mathrm{sl}(n)$.) Equivalently, $\mathfrak{k}$ is the intersection of a positive parabolic subalgebra (omit some of the negative roots) and of a negative parabolic; A. Joseph [9] has therefore also called them "biparabolic". For seaweed subalgebras of $\mathrm{sl}(n)$, which include all $\mathcal{P}(n,m)$, Dergachev and Kirillov define a functional F_S the importance of which is that it is always regular. As mentioned above, The Dergachev-Kirillov functional is constructed by taking into S elements on antidiagonals starting at the corners and proceeding until one reaches the main diagonal. For $\mathcal{P}(n,m)$ it consists of all pairs $(i,j) \in \Pi(n,m)$ with $i \neq j$ and either $i + j = m + 1, i > j$ or

$i+j=n+1, i<j$ or $i+j=n+m+1, 1\geq n-m$. (The first of these three sets will be empty for $m=1$ and the last will be empty for $m=n-1$; the diagram for $\mathcal{P}(7,3)$ is in Section 3.) Since this functional is always regular, when $(n,m)=1$ it is Frobenius and small in our sense. While there is yet no general closed formula for the index of a seaweed algebra, Dergachev and Kirillov give an algorithm, using a 'meander' built from their S, which once the omitted roots are specified rapidly computes the index. This meander is the graph $\gamma(S)$, which for their S is always a disjoint union of loops and chains (no branching); the index is 2(#loops)+(#chains)+(#isolated vertices)−1. For $\mathcal{P}(n,m)$ with $(n,m)=1$ it is easy to verify that the graph $\gamma(S)$ is a single chain (with possible reversals of arrows), giving an index of zero, but $\Gamma(S)$ generally has branches. The automorphism of $\mathrm{sl}(n)$ carrying $\mathcal{P}(n,m)$ to $\mathcal{P}(n,n-m)$ carries the Dergachev-Kirillov functional of the former to that of the latter.

11 The Belavin–Drinfel'd solutions to the MCYBE

Belavin–Drinfel'd have given an explicit construction of all solutions to the MCYBE associated to a simple Lie algebra $\mathfrak{g}$ in [2]. We do not need a full description of their work in what follows; the reader is referred to [2] for details. We shall use the fact, however, that the set of solutions is a finite disjoint union of components each of which is determined by an "admissible triple", which is a bijection between two subsets of positive simple roots of $\mathfrak{g}$, satisfying certain properties.

For the case $\mathfrak{g}=\mathrm{sl}(n)$ the simple roots may be identified with the set $\{1,2,\ldots,n-1\}$ and an admissible triple $\mathcal{T}$ is in effect a bijection $T:S_1\to S_2$ between subsets of $\{1,\ldots,n-1\}$ such that (1) for every $i\in S_1$ there is an r with $T^r(i)\notin S_1$ and (2) T preserves adjacency, i.e., if $i,j\in S_1$ with $|i-j|=1$ then $|T(i)-T(j)|=1$. There is a natural partial order among triples where $\mathcal{T}=(S_1,S_2,T)\prec\mathcal{T}'=(S_1',S_2',T')$ if T is the restriction of T' to some subset S_1 of S_1'. Any admissible triple with $\#S_1=n-2$ is then maximal in the partial order of triples, and these were all determined in [7]: Denoting the omitted element of S_1 by $n-m$, it must be the case that m and n are relatively prime (suggesting a relation with the fact that $\mathcal{P}(n,m)$ is then Frobenius); the omitted element of S_2 is m, and T sends every $i\in S_1$ to $i+m$ understood modulo n. With the standard triangular decomposition, $\mathrm{sl}(n)=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$, each solution to the MCYBE is of the form $\gamma+\beta+\alpha$ where $\beta\in\mathfrak{h}\wedge\mathfrak{h}$ and $\alpha\in\mathfrak{n}^+\wedge\mathfrak{n}^-$ are determined by $\mathcal{T}$ and $\gamma=\sum_{i<j}e_{ij}\wedge e_{ji}$. In particular α is uniquely determined by $\mathcal{T}$ and so we write it as $\alpha(\mathcal{T})$.

Let $r'=\gamma+\beta+\alpha(\mathcal{T}')$ be a solution to the MCYBE associated to some triple $\mathcal{T}'$. If $\mathcal{T}\prec\mathcal{T}'$, then it was asserted in [8] that $r=\gamma+\beta+\alpha(\mathcal{T})$ is in the closure of the orbit of r' under the operation of $SL(n)$. We show this here. (The idea of the proof probably works for all simple $\mathfrak{g}$.) If $h\in\mathfrak{h}$ then $[h,\gamma]=[h,\beta]=0$, so it is sufficient to find an h such that $\exp(th)\alpha(\mathcal{T}')\exp(-th)\to\alpha(\mathcal{T})$ as $t\to\infty$. Now $\mathcal{T}'$ establishes a partial order on the roots which we can extend to a linear order, and by renumbering we may suppose that this coincides with the

natural order $1, \ldots, n-1$. If $h \in \mathfrak{h}$ has diagonal entries $h_1, \ldots, h_n$ then setting $\lambda_i = h_i - h_{i+1}, i = 1, \ldots, n-1$ one has $[h, e_{i,i+1}] = \lambda_i e_{i,i+1}$. Since α' will now be a sum of elements of the form $e_{i,i+1} \wedge e_{j,j+1}$ with $j > i$, $\exp(th)\alpha(\mathcal{T}')\exp(-th)$ will have a finite limit as $t \to \infty$ whenever $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$. To make the limit agree with $\alpha(\mathcal{T})$ it is sufficient to make some of these inequalities strict, and it is trivial to find an $h \in \mathfrak{h}$ with these properties. Interestingly, in all examples studied, if we begin with $P(n,m)$, let F be the cyclic functional and take $h = \hat{F}$, the associated principal element, then this h has the required properties with respect to the associated $\mathcal{T}'$. Moreover, for the principal element h, we have $\exp(th)\alpha(\mathcal{T}')\exp(-th) \to \alpha(\mathcal{T})$ as $t \to \infty$ where T omits from T' only certain mappings sending an ith root to a jth root with $j < i$. In general it does not remove all of these except when there is just one. For example, with $P(5,2)$ where the progression of roots is $1 \to 3 \to 2 \to 4$ the map from 3 to 2 is removed, this being the only mapping in "reverse direction". However, with $P(8,5)$ where the progression of roots is $5 \to 2 \to 7 \to 4 \to 1 \to 6 \to 3$, only the mappings $5 \to 2$ and $4 \to 1$ are removed while $7 \to 4$ and $6 \to 3$ remain.

12 Local rings associated to a graph; a reconstruction theorem

This section is added to show that questions about graphs can be expressed as ones in local algebra (but that does not necessarily make them any easier). To every graph we associate two local rings, "full" and "reduced". The first captures all the information in the graph with but one exceptional case.

Suppose that we have a graph Γ whose sets of edges and vertices will be denoted by E and V, respectively. In the polynomial ring $K[E]$ generated by the edges, let I be the ideal generated by all relations of the form $e_1 e_2 = 0$ whenever $e_1, e_2 \in E$ have a common vertex and set $K\Gamma = K[E]/I$; we will call it the *full local ring of* Γ. Since I is homogeneous, $K\Gamma$ continues to be graded. Denote its radical (augmentation ideal) by J. Then J is nilpotent, and its index of nilpotence is $\mathrm{mn}(\Gamma) + 1$. The dimension of J^k is the number of disjoint k-tuples of mutually disjoint edges in Γ. For example, if Γ is a square then $\dim J = 4$ (the number of edges) and $\dim J^2 = 2$. One also has $\mathrm{mn}(\Gamma) = 2$ and $J^3 = 0$. Knowing $K\Gamma$ does not determine Γ since the triangle and three-pointed star both have local ring isomorphic to a ring $\mathcal{R}_3$ generated over K by three variables all of whose squares and products vanish, but we will show that amongst connected graphs this is the only counterexample. The triangle and the three pointed star are clearly the only graphs with $\mathcal{R}_3$ as local ring since the radical has dimension three, so there are exactly three edges, and the three-pointed star and triangle are the only configurations in which each has a vertex in common with every other. The ring can not recognize that the triangle has only three vertices while the star has four. One can tell from $K\Gamma$ if Γ is connected. Call a local ring *graph connected* if its radical J can not be written as a direct sum of subspaces J_1 and J_2 such that $x_1 \in J_1, x_2 \in J_2$ and $x_1, x_2 \neq 0$

imply $x_1 x_2 \neq 0$. (There are other concepts of connectedness.) It is evident that a graph is connected if and only if its full local ring is graph connected.

Theorem 7 *A connected graph* Γ *can be reconstructed from* $K\Gamma$ *except when* $K\Gamma \cong \mathcal{R}_3$*; connected graphs with isomorphic full local rings are isomorphic except for the triangle and the three-pointed star.*

PROOF. We show that one can reconstruct a graph Γ from its full local ring $K\Gamma$ as long as the ring is graph connected and the dimension of its radical J is not three. The cases where $\dim J$ are smaller than three are trivial, so we may assume that it is at least four. A *zero algebra* is one in which all products vanish. If the graph Γ contained no triangles then we could reconstruct it easily from $K\Gamma$. The vertices would then be in one-one correspondence with the maximal zero subalgebras of $K\Gamma$ and the vertices represented by two zero subalgebras Z and Z' would then be joined by an edge exactly when $Z \cap Z' \neq 0$, in which case the intersection would have dimension one. (Note that having $Z \cap Z' \neq 0$ is equivalent to the existence of a non-zero element which annihilates both, in which case there must also be non-zero elements in Z and Z', respectively, whose product is not zero, else Z would coincide with Z'.) The only problematic case is that where the maximal zero algebra Z has dimension three, i.e., is isomorphic to the radical of $\mathcal{R}_3$, for then we can not tell immediately if it has come from a triangle which is a subgraph of Γ or a vertex whose star is the three pointed star. Since the dimension of the radical is greater than three there must be some maximal zero subalgebra Z' with $Z \cap Z' \neq 0$, else the local algebra would not be graph connected. For in the notation above we could then take $J_1 = Z$ and J_2 to be the sum of all other zero subalgebras of the radical. If $Z \cap Z'$ has dimension one then Z and Z' represent vertices. The only other possibility is that the dimension is two, in which case Z came from a triangle and Z' from one of its vertices; that vertex is determined by the particular two dimensional subspace $Z \cap Z'$ of Z. $\square$

Let $\bigwedge V$ be the exterior algebra generated by the vertices V of Γ and $\bigwedge_{\text{even}}$ be its even subalgebra, i.e., the subalgebra generated by the unit element and all elements of the form $v \wedge v'$. Now assign an arbitrary orientation to every edge e of Γ so that we can say which vertex is initial and which terminal. Then we can define a ring morphism $K\Gamma \to \bigwedge_{\text{even}}$ by sending every edge e with initial vertex v and terminal vertex v' to $v \wedge v'$. The image $(K\Gamma)_{\text{red}}$ is the *reduced local ring*. It does not depend on the choice of orientation, is naturally graded, and is again a commutative local ring whose radical still has index of nilpotence equal to $\text{mn}(\Gamma) + 1$, but its dimension over K may be smaller than that of $K\Gamma$. For example, if Γ is a square with sides (in counterclockwise order) x, y, z, w and respective vertices a, b, c, d, then except in the case of characteristic 2, J^2 has dimension 2, being spanned by xz and yw, while its image has dimension only 1, being spanned by $a \wedge b \wedge c \wedge d$.

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